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ROBUSTNESS IN OPTIMUM DESIGN OF FREEFORM MECHANICAL PARTS

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ABSTRACT

A method to identify robust designs of mechanical parts with free-form shapes is proposed. For each design, the geometry and operating conditions represent one design point in the design space, with noise altering the design point leading to a change in performance. A shape optimization process is conducted for each example problem. Each successive iteration during the process produces an iterative design point with the final one being the optimum design. Once the process is completed, a design of experiment approach is used to apply noise in order to generate samples around each and every iterative design point. Then a simple statistical method is utilized to analyze the samples in order to evaluate the robustness of each iterative design. The results show that an optimum design is not necessarily robust.

1. INTRODUCTION

An earlier study (Cholaseuk et al., 1999) explored the optimum design of fluid flow devices using designed numerical experiments. The objective function for optimization took the form of a performance measure, such as the pressure drop across a diffuser. The geometric shape of the device was varied in order to maximize this objective function. In this paper we explore the stability of such designs.

Engineers refer to the stability of designs as the robustness of designs. Simply stated, a robust design is one that delivers roughly the same performance in the presence of inevitable variations in the manufactured instances of a product as well as in its operating conditions. A mathematical abstraction of the notion of robustness is shown in Fig. 1. A point p in the space P of product and environmental parameters maps to a point q in the space Q of performance indicators. Assuming some mild smoothness of this mapping function, a neighborhood $N_P(p)$ maps to a neighborhood $N_Q(q)$. The neighborhoods are shown as shaded regions in Fig. 1. A robust design is a point p in P whose finite neighborhood $N_P(p)$ maps to a "small" neighborhood $N_Q(q)$. In contrast, an optimum design is a point p in P whose mapping q in Q achieves the maximum of the desired performance over Q.

Designers seek optimum designs that are also robust. If this is not achievable, then suboptimal designs that exhibit robustness may be acceptable. But any design, optimal or otherwise, that is not robust is not an acceptable engineering solution. This opens up several interesting questions. What is the measure of smallness of the neighborhood $N_Q(q)$? How can one explore the neighborhood of points in P? We propose to answer these questions in this paper and test them on several examples. We make no special assumptions about the mapping function F except that we have at our disposal a function evaluator that, given a point p in P, evaluates a point q in Q. We use computer simulation to perform this evaluation. We then sample several discrete points in P and probe the neighborhood of each point for robustness.

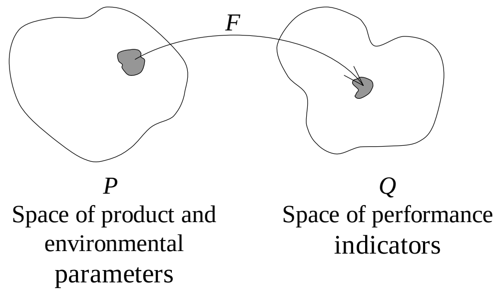


Figure 1. A mapping for a mathematical abstraction of robustness in design.

A brief literature survey is presented in section 2. A sampling scheme that chooses a discrete set of points in P is described in section 3. Subsequently four example problems to which the present approach is applied are presented in section 4. These are

(i) Minimization of the transverse velocity at the inlet and outlet of a ninety-degree elbow in potential flow,
(ii) Maximization of a pressure rise through a plane laminar flow diffuser with fixed inlet width and length,
(iii) Minimization of the weight of a cantilever beam with end load and with a constrained stress limit and
(iv) Minimization of the weight of a torque arm with a constrained stress limit.

The results are analyzed and discussed in section 5 and summarized in the last section.

2. LITERATURE SURVEY

Researches have adopted a variety of approaches in the area of computational shape optimization of mechanical parts. These include black-box optimization (Cholaseuk et al., 1999), adjoint operator or optimal control based methods (Pironneu, 1974, 1984, Cabuk and Modi, 1992) and genetic algorithms (Richards, 1995). With modeling approximation and limited solver accuracy, optimum shapes obtained using computational shape optimization can only be approximate. Moreover, in practice the mechanical parts may be subjected to uncontrollable factors (or noise) such as finite manufacturing tolerances and variations in operating conditions. In order that a chosen design remains close to optimal during practical use, the design must be relatively insensitive to noise. A design that meets these qualifications will be referred to as a robust design.

Phadke (1989) suggests one way to classify noise by its source as internal noise, external noise and deterioration. Unit-to-unit variation due to manufacturing imperfections such as dimensional tolerances or variations in material properties is internal noise. External noise consists of changes in operating conditions such as temperature, humidity etc. Deterioration refers to changes of the parts from theirs original state in time due to aging and wear. In this study, the source of noise is not of concern and hence deterioration will be considered as part of internal noise.

A robust design, by definition, requires the minimization of the variance of performance while the goal of an optimum design is to maximize or minimize the mean value of the target quantity. To achieve robustness and optimality at the same time may not always be possible in most cases and a compromise must be made. Following Taguchi (1986), the quantity signal-to-noise ratio (*SNR*) is used in robustization. It characterizes the ratio of mean performance to variance of performance under the presence of noise. Three formulations of *SNR*'s for different objectives are shown in Eq. (1) to (3).

$$\text{Minimization:}\quad SNR_S = -10Log\left(\frac{1}{m}\sum_{i=1}^{m} q_i^2\right) \tag{1}$$

$$\text{Maximization:}\quad SNR_L = -10Log\left(\frac{1}{m}\sum_{i=1}^{m}\frac{1}{q_i^2}\right), \tag{2}$$

$$\text{Target-is-best:}\quad SNR_N = 10Log\left(\frac{1}{m}\cdot\frac{\overline{q}^2}{s^2}\right), \tag{3}$$

where q_i (i=1,2,3,…, m) is the performance of neighboring samples around the design point and represents the effects of noise. Moreover mean performance $\overline{q}$ and variance s^2 are:

$$\overline{q} = \frac{1}{m}\sum_{i=1}^{m} q_i\ , \tag{4}$$

$$s^2 = \frac{1}{m-1}\sum_{i=1}^{m}(q_i - \overline{q})^2\ , \tag{5}$$

where s is standard deviation.

To compute *SNR*, noise is applied to each design point. The performance of each of the m neighbors (due to noise) is evaluated. Note that the following quantities are known as part of design specifications: target range of performance, range of operating conditions (external noise) and allowable tolerances (internal noise). In practice, noise occurs in unpredictable patterns, so there is no obvious way to simulate noise. Random or systematic patterns such as design of experiment (full or partial factorial) are the popular choices for noise simulation. Figure 2 shows the neighbors of the design point obtained by applying 2^2 combinations of two noise factors.

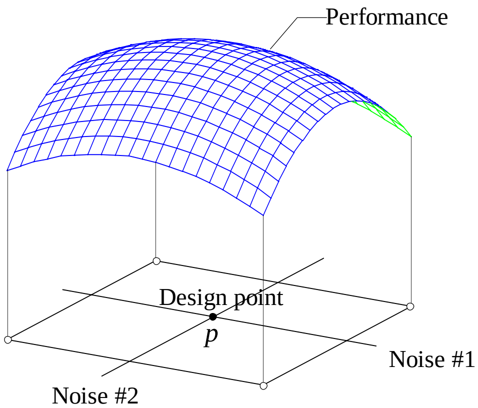


Figure 2. Neighborhood samplings of the design point.

From Eq. (1) to (3), a larger *SNR* should lead to a design that is both robust as well as close to the optimum; so a process that maximizes *SNR* would be a good means to achieve a compromise. However, there are some disadvantages in the use of *SNR*. First, it requires numerous direct solutions in order to simulate the effect of noise. If we use a design of experiment method to robustize a design with the *n1* design variables and consider *n2* noise factors, we must construct crossed arrays (Myers and Montogomery, 1995). Figure 3 shows a crossed array with *n1*=2 and *n2*=3. The filled circles represent the inner array of design variables used to estimate derivative information (of *SNR* with respect to the design variables). The empty circles refer to the outer arrays that take into account the effect of noise to compute *SNR* for each point of the inner array. A total of $2^{n1} \times (2^{n2}+1) = 36$ direct solutions are needed for one design iteration. Another disadvantage is an aliasing problem, since mean and variance are cofounded in *SNR*. For example, to maximize the performance, a design with high mean and high variance can have the same *SNR* as another design with low mean and low variance. In such cases, constraints on mean, variance and range of performance have to be imposed in the optimization process to determine a feasible region. An example of a problem one can encounter in using *SNR* can be found in Wilde (1991).

With the mean and variance that are computed at the same time as *SNR*, and the specification of performance range, one can also use the Normal distribution

$$G(q) = \frac{1}{\sqrt{2\pi}\, s} e^{-\frac{1}{2}\left(\frac{q-\bar{q}}{s}\right)^2}, \qquad (6)$$

where $\bar{q}$ is the mean and s is the standard deviation, to determine the probability of failure which is one more parameter to be concerned in robust design.

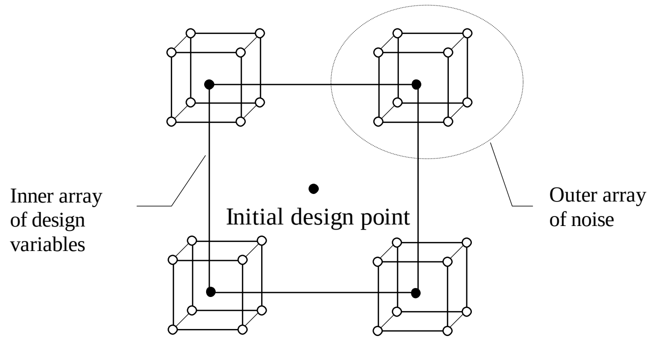


Figure 3. Crossed arrays for robustization.

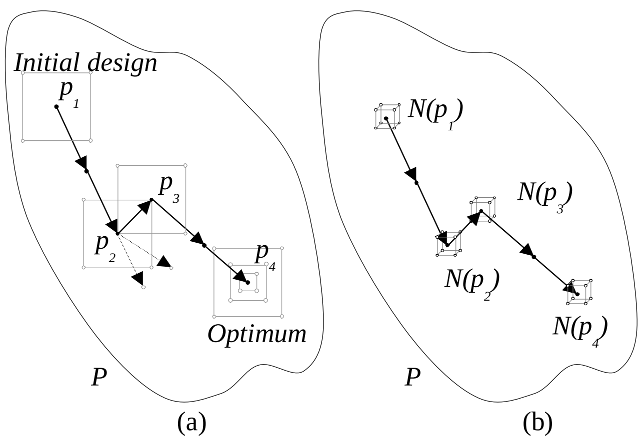
Other than the use of *SNR*, Cagan and Williams (1993) propose a method based on an extension of the Lagrangian and *KKT* conditions of optimality to take into account the measurement of flatness and curvature of the objective function. The method requires the use of second order derivatives and finally human judgement may be required. There are several other techniques for robust design such as *worst case analysis*, *corner space evaluation*, etc. A recent article by Chen and Du (1999) provides a comparison of some of these methods.

3. METHODOLOGY FOR ROBUSTNESS TESTING

Most applications of robust design techniques have been to problems with analytical solutions permitting inexpensive computation. In the present study, we focus on the design of freeform shapes, which involve many design variables and where no prior analytical solution to the relationship between performance and shape is available. Moreover methods such as *worst case analysis* may not be appropriate in our applications since the worst case noise factors are not necessarily the extreme values of the specifications. And due to a large number of design variables and time consuming yet inaccurate iterative direct solution, we cannot afford to use *SNR*'s as objectives of the designs or the use of Lagrangian based methods. Our approach to the problem is to first perform a numerical shape optimization process to maximize (or minimize) the performance. Then check for robustness of each iterative design.

To perform the shape optimization we use the same strategy as in our former study (Cholaseuk et al., 1999). The solvers used to determine the numerical values of the objective functions are treated as black boxes. The portions of the shapes that are to be optimized are represented with cubic B-spline curves (Hoschek and Lasser, 1993). Fixed geometrical constraints can be applied to the shape by fixing the appropriate control points. Coordinates of the "free" control points become the design variables. Design of experiment method (Box et al., 1979) is used to conduct the search and find the steepest direction (gradient) to adjust the design variables. The number of control points is progressively increased as necessary during the iteration process. The pattern of the search for two design variables is shown in Fig. 4(a).

Once the optimum shape is obtained in each problem, the robustness of that shape is explored at each design iteration by applying noise (internal or external or both) to each critical design point in Fig. 4(b). A total of 2^n factorial combinations of noise are applied to the design to generate the neighborhood samples. Direct solution for performance evaluation is obtained for each sample. Then we compute *SNR*, mean, standard deviation and range of the performance for each iterative design. Using a Normal distribution model, we also estimate the probability of failure for each design.



These parameters are compared at each design iteration and the most robust design for each problem is identified.

Figure 4. (a) Search pattern in the design space P for optimum design, (b) Applying noise to critical points.

Noise in a free form shape can be viewed as changes of shape within a tolerance envelope of the size ±*d* as shown in Fig. 5(a). There are infinite number of free form shapes in the envelope. In our optimization algorithm, the shapes are represented by B-spline curves. The number of control points that define the curve is progressively increased during the optimization process until no further improvement in objective function is achieved. As a result, the design from each successive iteration may be represented with different number of control points. Since the control points are treated as design variables subject to noise and since the number of neighbors for each design point are given by 2^n factorial (for n free control points), the number of neighbors will be different at each design iteration. The varying number of neighbors makes it difficult to compare *SNR*'s, standard deviations and means obtained at design iterations. Moreover, in practice, noise in free-form shape mechanical parts does not originate at the designed control points but at the parts themselves.

To address this problem, once the optimization process is terminated, the control points themselves are discarded and only the shapes obtained from design iterations are considered. Applying the changes directly to the shape generates noise on each shape. We impose new quadratic B-spline curves on the shape as noise. Figures 5(b) and (c) show different shapes obtained by imposing different quadratic B-spline curves over the designed shape. With this approach we can control the number of neighbor samples by controlling the number of the B-spline control points. With the same numbers of samples, the *SNR's* from different design iterations are comparable. As more control points are used, more shape variations within the tolerance envelope are obtained, leading to a better estimate of robustness. From the convergence study (shown in the discussion), we chose to use 5 free control points (32 samples per design).

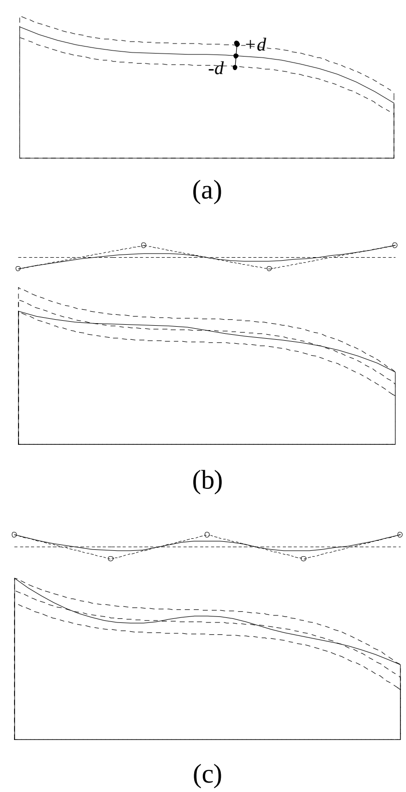


Figure 5. (a) Designed shape and tolerance envelope, (b) 4-control point quadratic B-spline curve overlapping the designed shape, (c) 5-control point quadratic B-spline curve overlapping the designed shape.

4. EXAMPLE PROBLEMS

Ninety-degree elbow in potential flow

An idealized problem of a two-dimensional potential flow through a ninety-degree elbow is considered. This problem does not account for the effects of viscosity, nor does it allow three-dimensional behavior. However it provides a test bed for validating the concepts of optimization and robust design. With a potential flow assumption, the governing equations for fluid flow are reduced to the Laplace equation. A boundary element method (Liggett and Lui, 1983) is used for flow solution after discretizing the boundary into 76 segments. The objective is to find the shape of the inner wall that minimizes the transverse velocity components at the inlet and outlet sections. Hence the objective is to minimize the sum of the modulus of the transverse velocity component at the inlet and outlet boundaries and is given by

$$\min\ q \text{ where } q = \sum_i \left| \frac{d\psi_i}{d\vec{n}} \right|, \tag{7}$$

i = inlet and outlet boundary nodes.

For the purpose of examining robustness internal noise will be generated within a shape tolerance of ±0.01. There is no external noise introduced in this case. The optimum shape and some intermediate shapes are shown in Fig. 6. The optimum shape is obtained at the eighteenth design iteration.

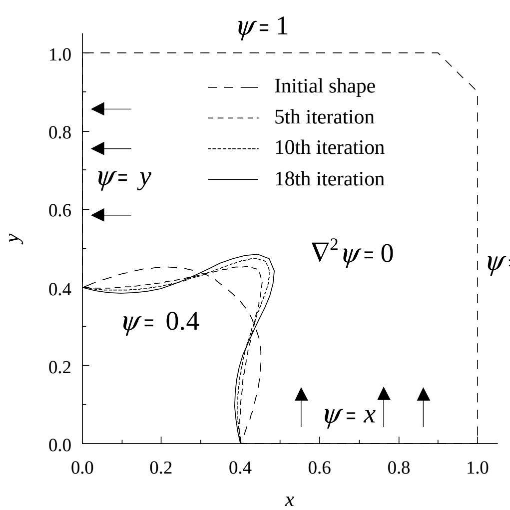


Figure 6. Evolution toward the optimum shape.

At each design iteration the signal to noise ratio, SNR_S and the standard deviation s is obtained from Eq. (1) and (5). History of convergence of the objective function, SNR_S and standard deviation are shown in Fig. 7.

The optimum design has an objective function value of 0.3 compared to 2.05 for the initial design. The optimum shape is found to be robust- the indicators being high SNR_S, low standard deviation and small range of performance variation. We do not specify the performance range for this particular example, so we do not look at the probability of failure here.

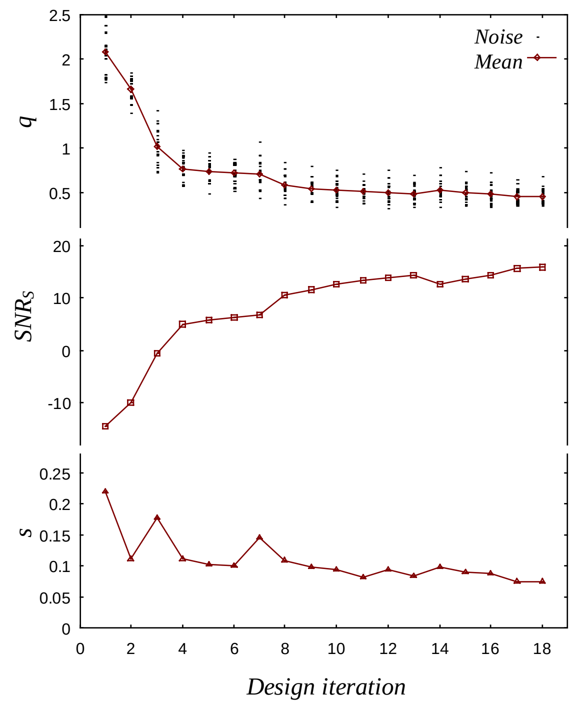


Figure 7. Variation of objective function mean along with the noise on q, SNR_S and s with design iteration.

Plane diffuser in laminar flow

The second problem considered is the design of a plane symmetric diffuser (symmetric half is shown in Fig. 8) for an incompressible steady laminar flow governed by the Navier-Stokes equations. A flow solver described in Ferziger and Peric (1996) is used to solve for the flow fields. The number of control volumes used in the diffuser section is 30×10. Pressure rise in the diffuser is a function of diffuser geometry and the inlet flow parameters. We try to find a diffuser profile that maximizes the pressure coefficient, C_p.

$$C_p = \frac{P_{out} - P_{in}}{\frac{1}{2}\rho u_{in}^2}, \tag{8}$$

where P_{in} and P_{out} are the area-averaged pressures at diffuser inlet and outlet, u_{in} is the mean inlet velocity and ρ is the fluid density. In addition, a worst case minimum value of C_p =0.42 is specified as a requirement for the optimum design. Internal noise is generated within a tolerance of ±0.05W. External noise is generated by changing the operating inflow Reynolds number of 100 by ±20%. The Reynolds number is defined as $Re = \dfrac{\rho u_{in} W}{\mu}$ where μ is the viscosity.

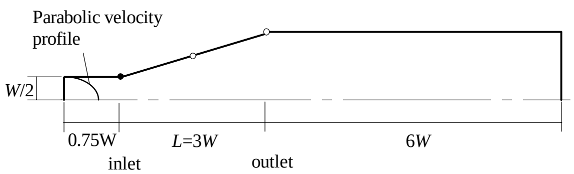


Figure 8. Plane symmetric diffuser.

The optimum diffuser and some intermediate designs are shown in Fig. 9. The optimum diffuser shape is obtained at the eleventh iteration. Convergence history of C_p, SNR_L and s are shown in Fig. 10. The dashed line for C_p represents the lower performance limit.

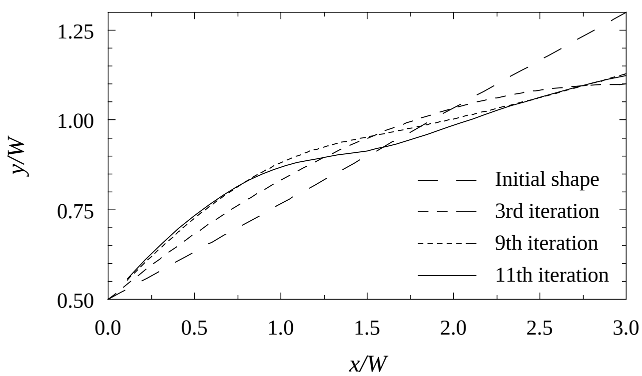


Figure 9. Evolution toward the optimum diffuser.

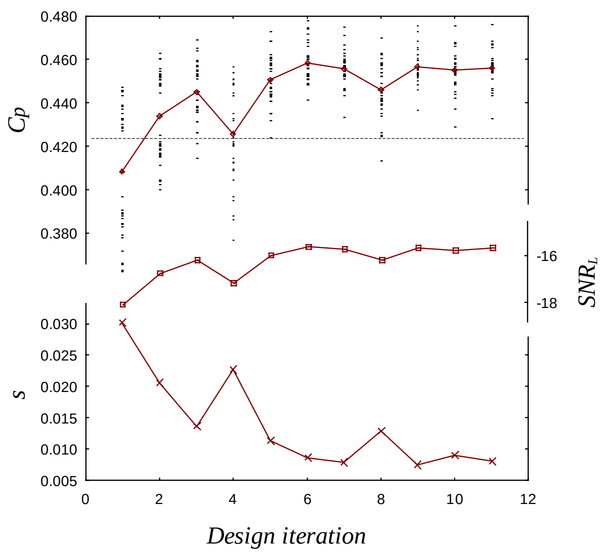


Figure 10. Variation of mean along with the noise on C_p, SNR_L and s with design iteration.

The initial diffuser produces a C_p value of 0.41 while the optimum diffuser (eleventh design) produces a C_p of 0.46. A mean C_p under noise is 0.456. We find that the optimum diffuser is robust even though the ninth design is found to be actually the most robust by a very small margin. The diffuser shapes are shown in Fig. 9 for comparison. In the laminar flow regime the Reynolds number also plays a role in determining diffuser performance; the pressure coefficient increases when *Re* decreases.

Cantilever beam

The optimization problem is to minimize weight of the cantilever beam of length L = 4 mm and thickness 1 mm, subject to an end load P = 10 N. Since the beam has a constant density, we set the objective function to be volume, V, of the beam. The material has ultimate stress σ_u = 120 MPa. With a safety factor of 1.2, the maximum design stress is set to be σ_d = 100 Mpa. To handle this stress constraint, a discrete penalty function is used. The objective function, f, is described in Eq. (9) where σ_e is the maximum element stress.

$$f = \begin{cases} V & \text{if } \sigma_e \leq \sigma_d \\ V \cdot (1. + \sigma_e / \sigma_d) & \text{if } \sigma_e > \sigma_d \end{cases} \tag{9}$$

An imposed geometrical constraint is that the beam be symmetric about the x-axis. Neglecting the weight of the beam, the analytical solution by beam theory is known to be a parabola represented by Eq. (10).

$$y = \sqrt{\frac{1.5P(L-x)}{\sigma_d}} \tag{10}$$

Internal noise is generated within a tolerance of ±0.05 mm. A finite element code (quadrilateral element) is used to solve for the stress in each of the 160 elements. The optimum shape and some intermediate shapes are shown in Fig. 11. The optimum shape is obtained at the eleventh design iteration.

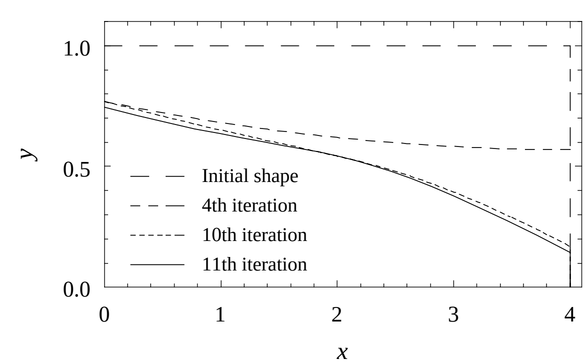


Figure 11. Evolution toward the optimum shape.

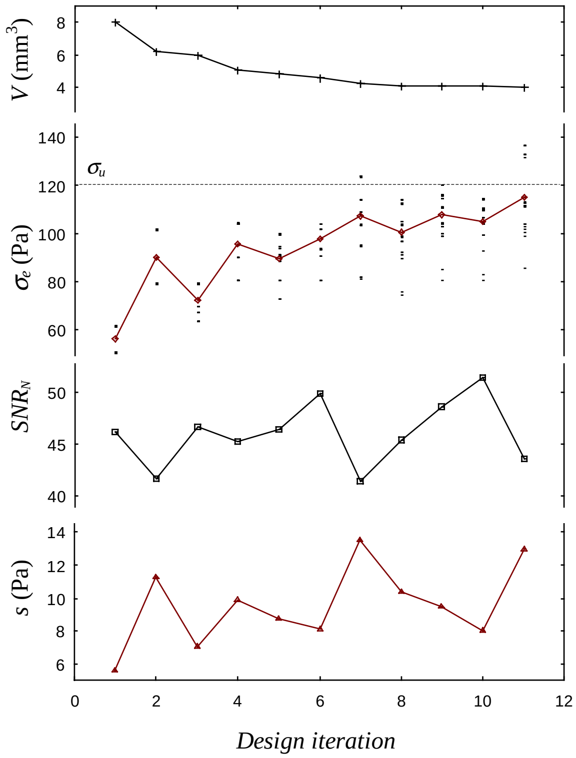


Figure 12. V, Variation of mean along with the noise on σ_e, SNR_N and s with design iteration.

Torque arm

A problem first introduced by Botkin (1982) where the objective is to minimize the weight of a torque arm subject to axial and transverse loads is considered next. Volume V is used as the objective function, since the density is constant. Ultimate stress of the material used is σ_u = 972 MPa. With a safety factor of 1.2, the design maximum stress is set to be σ_d = 810 Mpa. The shape of the straight boundary between the two holes is to be optimized. A stress constraint is handled by a discrete penalty function; objective function is described in Eq. (9). The overall geometry is shown in Fig. 13. The thickness of the arm is assumed to be 3 mm, radii of the small and large holes are taken to be 20 mm and 40 mm respectively. A geometrical constraint that the torque arm be symmetric about the x-axis is imposed. Internal noise is within the shape tolerance of ±5 mm. A finite element code quadrilateral element) is used to solve for the stress in the 282 elements.

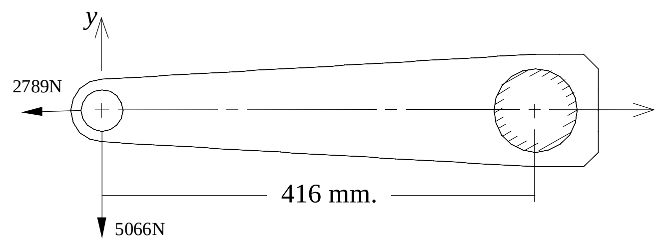


Figure 13. Torque arm.

The optimum shape and some intermediate shapes are shown in Fig. 14. The optimum shape is obtained at the thirteenth design iteration. Figure 15 shows the convergence history of V, σ_e, SNR_N and standard deviation. The last two parameters refer to element stress.

We may choose the design at the third iteration as a compromise between robust and optimum, because the third design has a much lower standard deviation and a volume of 74% of the initial design. This volume is not much larger compared to the minimum volume of 68% of the initial design for the thirteenth design. But if the ultimate stress of σ_u is of concern, all the designs are most likely to fail except the first design. This is because in designs other than the first design the means are located near or above the σ_u lines implying that there is a greater probability of failure.

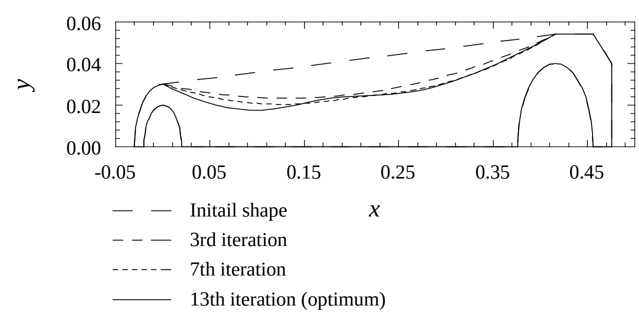


Figure 14. Evolution towards the optimum shape.

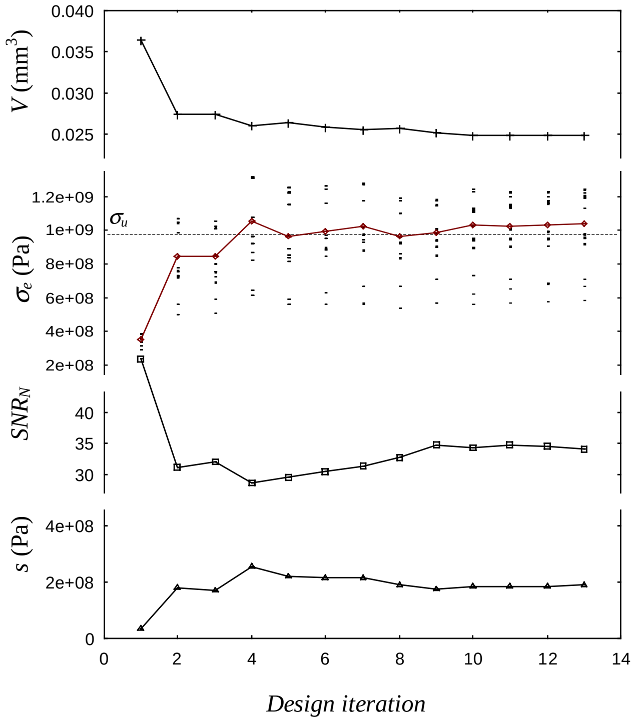


Figure 15. V, Variation of mean along with the noise on σ_e, SNR_N and s with design iteration.

5. DISCUSSION

We have compared the results obtained from the optimization process for all the four examples to results from other studies as shown in Fig.16(a) to (d).

Since the information we obtained from sampling is only a small part of the infinite neighborhood of points that lie around a design point, we can only predict the possibility of a design performing in the specified range. When looking at the distribution curves, we prefer a curve that is high and narrow (low deviation) and lies within the operating range. In the diffuser case, we have a very robust design as shown in Fig. 17. The probability of failure is close to zero. Figure 18 shows the distribution curves for the beam. From the beam example we choose the tenth iterative design as our robust design over the eleventh. From a normal distribution, there is 3% probability that the stress limit is exceeded in the tenth iterative design whereas the similar probability is 35% for the eleventh. If safety is however a greater concern, one might prefer to go with the fifth design, which has almost zero probability of failure but has more weight. The torque arm problem has the worst situation since most of the designs have a significant probability of failure, 23% for the third and 64% for the eleventh with only the initial design being really safe.

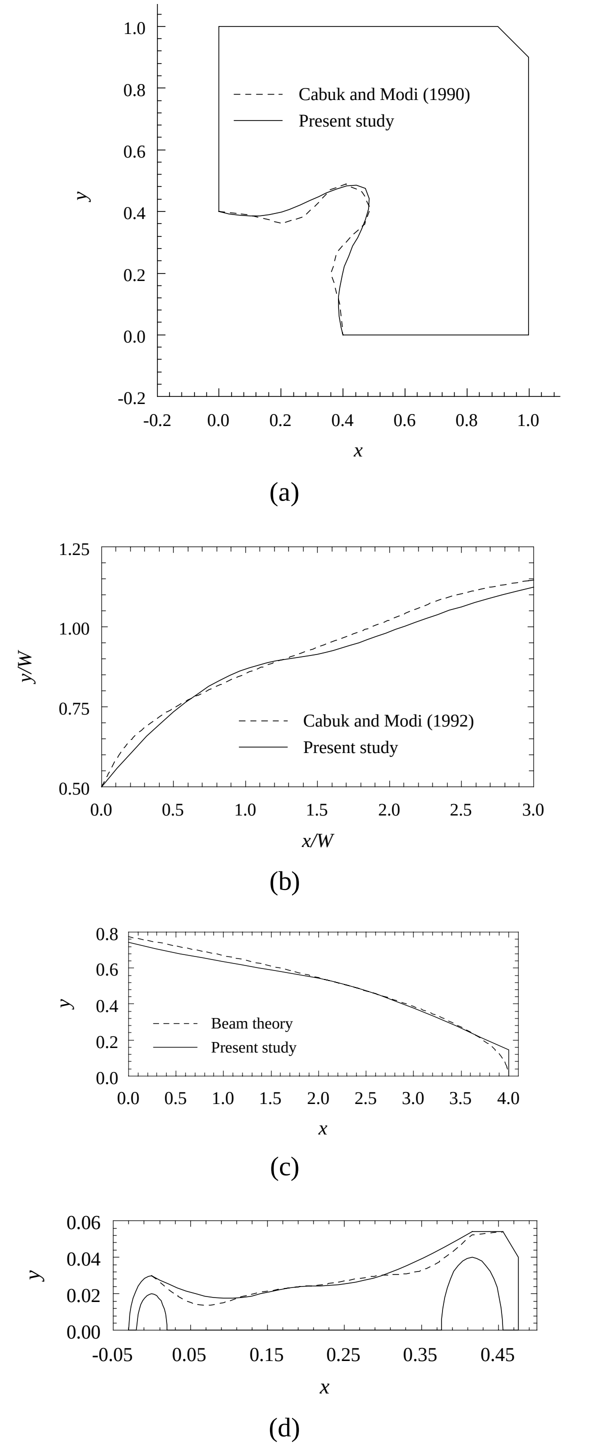


Figure 16. (a) Optimum 90-degree elbow compare to Cabuk and Modi (1990), (b) Optimum diffuser compare to Cabuk and Modi (1992), (c) Optimum cantilever beam compare to the result from beam theory (Eq. (10)), (d) Optimum torque arm compare to the result from Richards (1995) (dashed line).

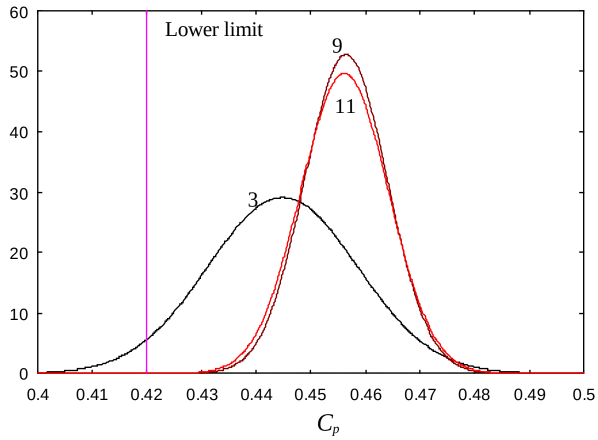


Figure 17. Normal distribution of C_p of some diffusers

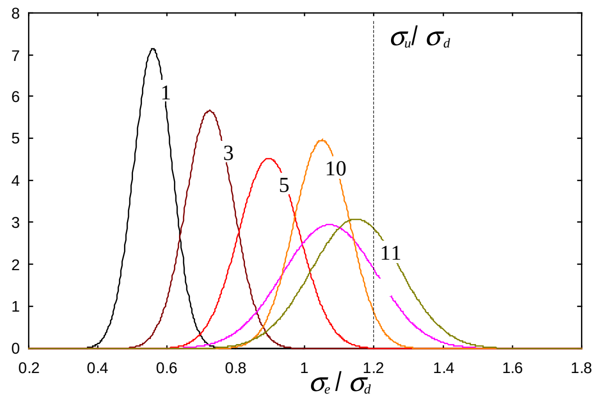


Figure 18. Normal distribution of stress of some cantilever beam designs

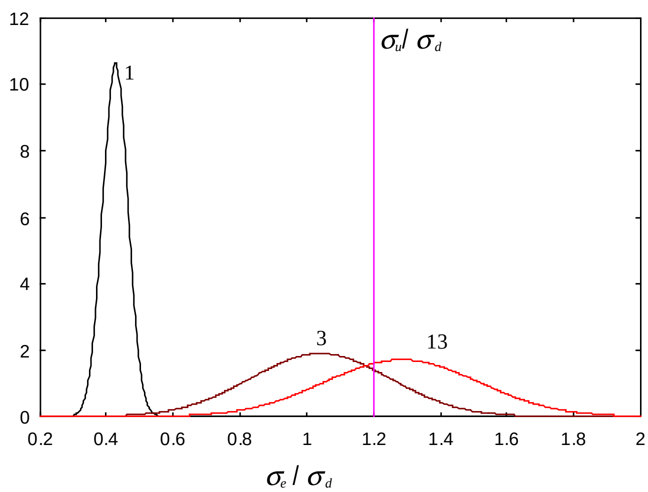


Figure 19. Normal distribution of stress of some torque arm designs

To determine the effect of the number of samples, a study was carried out by progressively increasing the number of control points that generate noise in the shape definition of the cantilever beam example. The number of samples increases as 2^n for n control points. The values σ_e, SNR_N and s for n=3,4,5 and 6 are shown in Fig. 20.

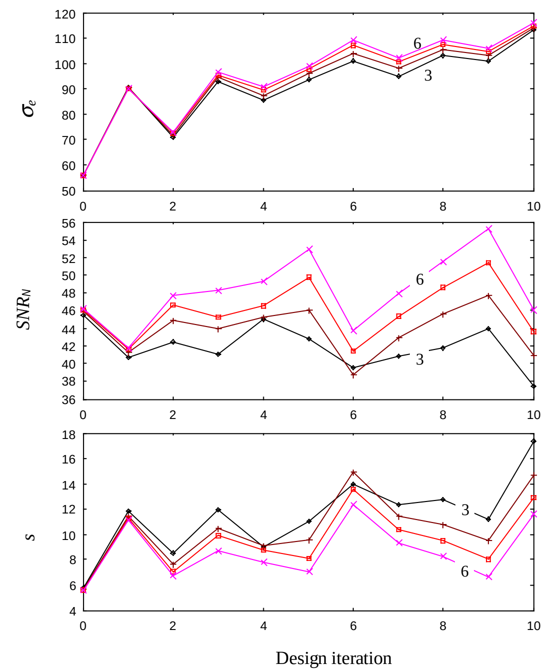


Figure 20. Effect of changing number of control points n for the beam problem

We find that *SNR* is not as useful in our problems, since the means and standard deviations provide clearer information for the selection of the robust design. And in three examples (Fig. 10, 12 and 15) *SNR*'s are just the reflection of the standard deviation, hence they give redundant information.

We observe that the robustness of the first two examples refers to objective functions that come from integration (sum of nodal transverse velocity in the first example and area averaged pressure in the second). In the last two examples, robustness is based on the stress constraint, which is the local maximum element stress. By nature, the values from integration are usually less sensitive to noise than the local point values. The physical nature of each problem also plays a role in robustness of the optimum results. For example, performance of the optimum diffuser seem to be not very sensitive to internal noise (shape changes) since for the optimum design, minor changes in shape do not cause much change to the whole flow field so the pressure rise does not change much. In contrary, for the torque arm problem, minor change of shape at some location can cause a large change in local stress concentration; hence the optimum torque arm is more sensitive to the internal noise.

6. CONCLUDING REMARKS

A generic *black box* optimization algorithm is used so that we do not have to modify the available solvers. The focus of the study is the examination of the robustness issue, leading to slower convergence of the optimization process.

In three of the cases considered, the only kind of noise applied is internal noise (tolerances of designed shapes). For the diffuser problem both internal and external noise were considered. External noise was in the form of a +/-20% Reynolds number variation. This variation was found to have a major effect on the diffuser performance.

The physical nature of the problems was found to play a major role in determining the robustness of the optimum results. The use of *SNR* was not sufficient for determining the robustness of a design. Instead robustness was judged by considering the mean, standard deviation, range of performance and probability of failure.

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